International Journal of Innovations & Research Analysis (IJIRA)
ISSN :2583-0295(Online), Impact Factor: 7.662, Volume 05, No. 03(I), July- September, 2025, pp 72-84

Nonlinear Dynamics of a Predator-Prey Model with Beddington-deAngelis Functional Response

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Citation:

ABSTRACT

This article uses Beddington and Holling type-II functional response to explore a tritrophic food web model system with prey-predator such as prey, midpredator, and top predator. This model's positivity, boundedness, local stability, and global stability were investigated. In addition, stability requirements are derived using the Routh-Hurwitz criterion. Both theoretical and numerical discussions of Hop bifurcation are included. Additionally, the Center Manifold Theorem has been used to establish the stability of non-hyperbolic equilibrium sites. Additionally, Matlab ode45 software has been used for numerical analysis, demonstrating the dynamic character of the model system.

***Keywords**: Prey-Predator System, Boundedness, Global Stability, Bifurcation, Center Manifold Theorem.*

Introduction

In ecology, interactions between predators and prey occur at higher trophic levels and predators can affect prey populations directly and indirectly [1, 2]. While thepredator's indirect influence involves the prey population, which has the potential toalter prey behavior, the predator's direct effect involves predating the prey [3, 4, 5].Functional response is a major aspect of the ecological model system. Functional responses that are used in many biological studies like Holling type I-IV, Ivlev, Crowly-Martin, Beddinton-DeAngelis, etc. Holling type II functional response is bounded and correct for biological systems which conclude that Holling type II functional response is more appropriate [6, 7, 8, 9, 10, 11]. Many researchers performed Holling type I functional response in higher order models [12, 13, 14, 15, 16]. For the higher order model Beddington–DeAngelis functional response is used. Many studies have discussed bifurcation analysis of the prey-predator system providing for prey refuge, and the existence of bifurcation (transcritical, Hopf). When one root is negative and the next is purely complex conjugate, this type of equilibrium point is non-hyperbolic. The stability of equilibrium point Center manifold theorem has been performed [17]. Group defense plays a major role in the dynamical system which plays an important part in prey- predator model system. When prey species are in large numbers, they have defense ability by creating herds, which shows decreased predation of the prey [18]. Some studies consider the impact of toxins which are harmful to many aquatic organisms in marine systems [19]. It is evident that in the process of algal blooms, algal aggregation plays a significant role. It has been found in past decades that the prey-predator stabilityhas been affected by toxic substance [20].Filter-feeding fish can reduce the algal bloom population which affects the healthy development of the marine system. Filter-feedingfish is widely applied in water bodies which is a direct method of manipulation to control cyanobacterial algal blooms [21].Some authors investigated the harvesting of prey and predators, when the density of the harvested population is large such harvesting increases to a limit value [22, 23]. Many authors have also included alternative food sources in their research and the behavior of a system is significantly influenced by substituting food sources for predators. When the density of favored prey is low, predators move to other meals [24, 25, 26].In this study, we considered the ecological terms logistic growth rate, consumption of prey by mid predator using Beddington function response, and consumption of midpredator by the top predator using Holling type II functional response. Beddington-DeAnglis function response and Holling type II functional response have not been used together. In this study, we illustrated the local and global stability of model system. Then, we performed the bifurcation by using constant c. Also, we found the stability using the Center Manifold theorem. Finally, theoretical results are found with some numerical analysis.

The importance of this article is:

- To introduce three-dimensional food chain prey predator model by using Beddington-DeAnglis functional response and Holling type II functional response.
    - To check the stability and bifurcation of the system.
    - To know the concept of Center manifold theorem.
    - To illustrate numerical analysis of the model system.

The article is given below: In part 2, model formulation of the non-spatial system has-been given. In part 2.1, dynamical behavior is analyzed. It is shown that equilibrium points exist then stability has been checked at each of the equilibrium points. Moreover, bifurcation of the model has been performed. In part 3, numerical simulations have been given. Finally, in part 4, discussion and conclusion has been given.

Motivation and Novelty

In our paper formulation of mathematical model based on three species food chain with prey refuge have been studied using Beddington DeAnglis and Holling type II functional responses. The Beddington-DeAnglis and Holling type II responses are especially helpful in modeling situations in which predators become more effective at increasing prey densities and less effective at lower densities. To improve the system's stability, the study combines these two functional responses together. This method can result in more resilient and stable ecosystem dynamics and enables a more sophisticated portrayal of predator-prey interactions. The attack rate of a generalist predator significantly influences the mathematical modeling of predator-prey dynamics. The attack rate directly determines the form and parameters of the functional response, which describes how the predator's consumption rate changes with prey density. Higher attack rate, increases the predator's efficiency in capturing prey, leading to a rapid decline in prey population. This can result in greater oscillations in population sizes and potentially destabilize the system if the prey population drops too low. Lower attack rate, decreases the predator's efficiency, allowing the prey population to grow, which might reduce oscillations and stabilize the system. Predator prey models demonstrate inhibitory effects in population dynamics, since the presence of predators slows the increase of prey populations. This interaction can keep the prey from growing too much and stabilize the populations. Also, the Center Manifold Theorem is a powerful tool in dynamical systems theory, particularly useful for analyzing the stability of non-hyperbolic equilibrium points. The Center Manifold theorem facilitates the analysis of a system's stability by lowering its dimensionality in the vicinity of a nonhyperbolic equilibrium point. Main target of this study is to introduce refuge ability and prevented predation of the prey. This model is completely new as it integrates the Beddington-DeAngelis functional response and the Holling type II functional responses, which has not been previously combined in ecological research.

The article is given below: In part 2 , model formulation of the non-spatial system has been given. In part 3, some definitions related to work are given. In part 4, dynamical behavior is analyzed. It is shown that equilibrium points exist then stability has been checked at each of the equilibrium points. Moreover, Global stability, effect of attack rate of generalist predator on specialist predator, bifurcation of the model, center manifold theorem have been performed. In part 5, numerical simulations have been given. Finally, in part 6, discussion and conclusion has been given.

System Model Formulation

Here, we consider three-dimensional interaction model. To propose the model system the assumptions are as below:

- Prey density is shown by U, mid-predator density is shown by V, and top predator density is shown by W.
- The prey species becomes larger with growth rate A1.
- K shows carrying capacity.
- Prey and mid-predator succeed Bedding ton functional response.
- Mid-predator and top predator succeed Holling type-II functional response.

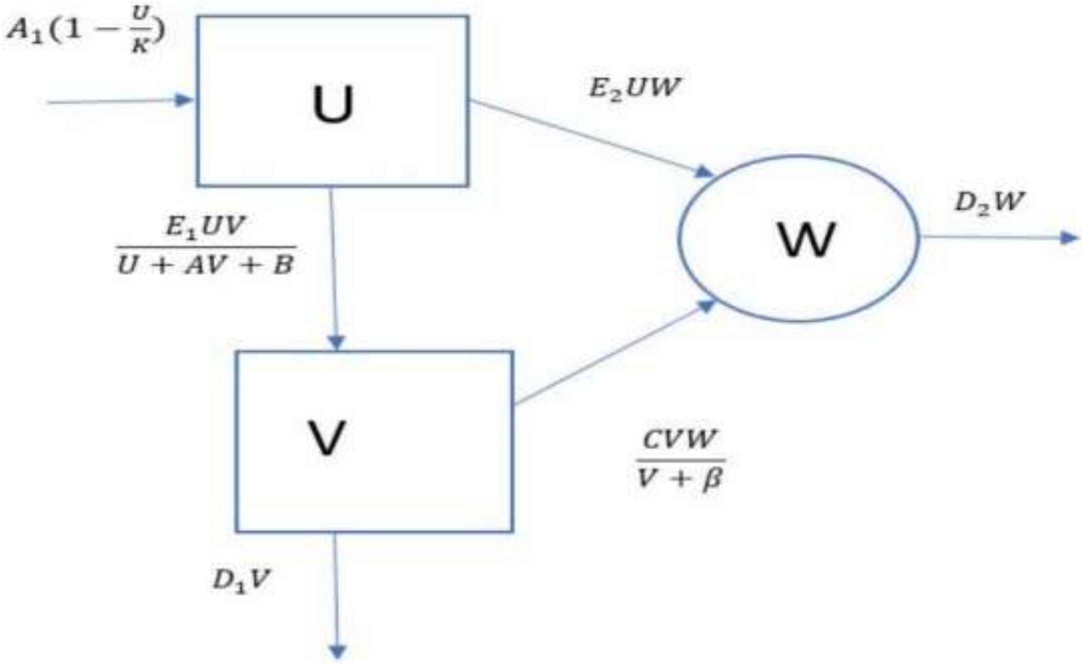

Fig. 1 Schematic diagram of model system

The model is expressed as below:

$$\frac{dU}{dT} = A_1U\left(1-\frac{U}{K}\right) - \frac{E_1UV}{U+AV+B} - E_2UW \tag{1}$$

$$\frac{dV}{dT} = \frac{\alpha_1E_1UV}{U+AV+B} - D_1V - \frac{C_1VW}{V+\beta} \tag{2}$$

$$\frac{dW}{dT} = \alpha_2E_2UW - D_2W + \frac{\alpha_3C_1VW}{V+\beta} \tag{3}$$

Subject to IC: $U(0)>0, V(0)>0, W(0)>0$. Now, we reduce the parameters, for this purpose, we put $u=\frac{U}{K}$, $A_1T=t$, $v=\frac{E_1V}{A_1K}$, $w=\frac{WE_2}{A_1K}$. The system turns as:

$$\frac{du}{dt} = u(1-u) - \frac{uv}{u+cv+\beta_1} - uw \tag{4}$$

$$\frac{dv}{dt} = \frac{\alpha_1uv}{u+cv+\beta_1} - m_2v - \frac{c_1vw}{v+b} \tag{5}$$

$$\frac{dw}{dt} = m_3uw - m_4w + \frac{m_5wv}{v+b} \tag{6}$$

$u(0)>0, v(0)>0, w(0)>0$. Here, $c=\frac{AA_1}{E_1}, \beta_1=\frac{B}{K}, m_2=\frac{D_1}{E_1}, b=\frac{\beta E_1}{A_1K}, m_4=\frac{D_2}{E_2}, m_5=\frac{\alpha_3C_1}{E_2}, m_3=\alpha_2K$.

Dynamical Behavior

In this part, we will discuss positivity, boundedness, stability, and bifurcation of the system.

Table 1. Ecological Parameters are described as:

Parameters	Meaning
A_1	Growth rate
E_1	mid predator's consumption rate on prey
A	Inhibitory effect
K	Carrying capacity
B	Half saturation constant
E_2	predator's consumption rate on prey
α_1	Conversion of prey biomass into mid predator
D_1	mid predator's death rate
C_1	Consumption rate of mid predator by top predator
β	Predator interference parameter
α_2	Constant rate of prey biomass into top predator
D_2	top predator's death rate
α_3	Constant rate of mid predator biomass into

- **Positivity**

Populations in the system never go extinct because of the positives we discovered. For this purpose, we integrate the equations by using ICs as below

$$u(t) = u(0)\left(\int_0^t \left[(1-u) - \frac{v}{u+cv+\beta_1} - w\right] ds\right) \quad (7)$$

$$v(t) = v(0)\left(\int_0^t \left[\frac{\alpha_1 u}{u+cv+\beta_1} - m_3 - \frac{c_1 w}{v+b}\right] ds\right) \quad (8)$$

$$w(t) = w(0)\left(\int_0^t \left[m_3 u - m_4 + \frac{m_5 v}{v+b}\right] ds\right) \quad (9)$$

Hence, if the initial conditions are non-negative then the right side is positive.

- **Boundedness**

In this part, we prove the boundedness of the system. It indicates the system is conducted in an appropriate manner.

Theorem 1: Solutions of the model system (1)-(3) is uniformly bounded in R_+^3.

Proof: Let solution of the model (1)-(3) is $(u(t), v(t), w(t))$.

Let us suppose a function $\psi(u, v, w) = u + \frac{v}{\alpha_1} + \frac{c_1 w}{\alpha_1 m_5}$, then we have $\frac{d\psi}{dt} = u(1-u) + \left[\frac{c_1 m_3}{m_5 \alpha_1} - 1\right] uw - \frac{w}{\alpha_1}\left[m_2 + \frac{m_4 c_1}{m_5}\right]$

Then we get $l > 0$, in a way that $\frac{d\psi}{dt} + \zeta\phi \le l$, which implies that: $\phi(t) \le \phi(0)e^{-\zeta t} + \frac{l}{\zeta}\left(1 - e^{-\zeta t}\right) \le \max\left(\phi(0), \frac{l}{\zeta}\right)$. Hence, the theorem is proved.

- **Equilibrium points and stability**

This analysis gives four equilibrium points (i) $E_1 = (0,0,0)$ (ii) $E_2 = (1,0,0)$ (iii) $E_3 = \left(\frac{m_4}{m_3}, 0, \frac{m_3 - m_4}{m_3}\right)$ (iv) $E_4 = (\tilde{u}_4, \tilde{v}_4, \tilde{w}_4)$

- **Stability analysis.**

In this part, first, linearize the system to get the stability then find the Jacobian matrix. Theorem 2: Eigen values of $J(E_1)$ is a saddle point.

Proof: Here $E_1 = (0,0,0)$. Then, $J(E_1)$ is $J(E_4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -m_2 & 0 \\ 0 & 0 & -m_4 \end{bmatrix}$

The eigenvalues of E_1 are $1, -m_2, -m_4$. Then, E_1 is a saddle point.

Theorem 3: E_2 is LAS if $\frac{\alpha_1}{1+\beta_1} < m_2$ and $m_3 < m_4$.

Proof: $J(E_2)$ is as follows:

$$J(E_2) = \begin{bmatrix} -1 & \frac{-1}{1+\beta_1} & -1 \\ 0 & \frac{\alpha_1}{1+\beta_1} - m_2 & 0 \\ 0 & 0 & m_3 - m_4 \end{bmatrix}$$

The eigen values of E_2 are $-1, \frac{\alpha_1}{1+\beta_1} - m_2$ and $m_3 - m_4$. Therefore, E_2 is LAS if $\frac{\alpha_1}{1+\beta_1} < m_2$ and $m_3 < m_4$

Theorem 4: $J(E_3)$ is LAS if $m_2 > \frac{\alpha_1\left(\frac{m_4^2}{m_3^2}+\beta\frac{m_4}{m_3}\right)}{\left(\frac{m_4}{m_3}+\beta_1\right)^2} - \frac{c_1(m_3-m_4)}{m_3 b}$ and saddle point if $m_2 < \frac{\alpha_1\left(\frac{m_4^2}{m_2^2}\beta\beta\frac{m_4}{m_3}\right)}{\left(\frac{m_4}{m_3}+\beta_1\right)^2} - \frac{c_1(m_3-m_4)}{m_3 b}$.

Proof: $E_3 = \left(\frac{m_4}{m_3}, 0, \frac{m_3-m_4}{m_3}\right)$, exists. Thus, the characteristic roots $J(E_3)$ is as given: $\frac{\alpha_1\left(\frac{m_4^2}{m_3^2}+\beta\frac{m_4}{m_3}\right)}{\left(\frac{m_4}{m_3}+\beta_1\right)^2} - m_2 - \frac{c_1(m_3-m_4)}{m_3 b}, \lambda_2$ and λ_3 Here, λ_2 and λ_3 are characteristic roots of the equation $\lambda^2 + \frac{m_4}{m_3}\lambda + \frac{m_4}{m_3}(m_3 - m_4) = 0$. Hence, $J(E_3)$ is LAS if $m_2 > \frac{\alpha_1\left(\frac{m_4^2}{m_3^2}+\beta\frac{m_4}{m_3}\right)}{\left(\frac{m_4}{m_3}+\beta_1\right)^2} - \frac{c_1(m_3-m_4)}{m_3 b}$ and saddle point if $m_2 \frac{\alpha_1\left(\frac{m_4^2}{m_3^2}+\beta\frac{m_4}{m_3}\right)}{\left(\frac{m_4}{m_3}+\beta_1\right)^2} - \frac{c_1(m_3-m_4)}{m_3 b}$

Theorem 5: E_4 is LAS, if conditions satisfy (i) A, B and $C > 0$, (ii) $AB > C$.

Proof: Now, for the equilibrium point $E_4 = (\tilde{u}, \tilde{v}, \tilde{w})$, exists. Thus $J(E_4)$ is as following:

$$J(E_4) = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}$$

where, $y_{11} = 1 - 2\tilde{u} - \frac{c\tilde{v}^2+\beta_1\tilde{v}}{(\tilde{u}+c\tilde{v}+\beta_1)^2} - \tilde{w}, y_{12} = \frac{-(\tilde{u}^2+\beta_1\tilde{u})}{(\tilde{u}+c\tilde{v}+\beta_1)^2}, y_{13} = 0, y_{21} = \frac{\alpha_1(\tilde{v}^2+\beta_1\tilde{v})}{(\tilde{u}+c\tilde{v}+\beta_1)^2}$, $y_{22} = \left[\frac{\alpha_1(\tilde{u}^2+\beta_1\tilde{u})}{(\tilde{u}+c\tilde{v}+\beta_1)^2} - m_2 - \frac{bc_1\tilde{w}}{(\tilde{v}+b)^2}\right], y_{23} = \frac{-c_1\tilde{v}}{(\tilde{v}+b)}, y_{31} = m_3\tilde{w}, y_{32} = \frac{bm_5\tilde{w}}{(\tilde{v}+b)^2} y_{33} = m_3\tilde{u} - m_4 + \frac{m_5\tilde{v}}{(\tilde{v}+b)}$.

The characteristic equation is as showing:

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \tag{10}$$

where, $A = -(y_{11} + y_{22} + y_{33})$

$B = y_{11}y_{22} - y_{12}y_{21} - y_{23}y_{32}$

$C = y_{11}(y_{22}y_{33} - y_{23}y_{32}) - y_{12}y_{21}y_{33}$

Hence, E_4 is LAS, as following conditions hold: (i) A, B and $C > 0$ (ii) $AB > C$.

Global Stability Analysis

Theorem 6: If we consider that

$$1-2\tilde{u}-\frac{c\tilde{v}^2+\beta_1\tilde{v}}{(\tilde{u}+c\tilde{v}+\beta_1)^2}-\tilde{w}>0 \quad (11)$$

$$\left[\frac{\alpha_1(\tilde{u}^2+\beta_1\tilde{u})}{(\tilde{u}+c\tilde{v}+\beta_1)^2}-m_2-\frac{bc_1\tilde{w}}{(\tilde{v}+b)^2}\right] \quad (12)$$

$$m_3\tilde{u}-m_4+\frac{m_5\tilde{v}}{(\tilde{v}+b)}>0 \quad (13)$$

$$\left[1-2\tilde{u}-\frac{c\tilde{v}^2+\beta_1\tilde{v}}{(\tilde{u}+c\tilde{v}+\beta_1)^2}-\tilde{w}\right]\left[\frac{\alpha_1(\tilde{u}^2+\beta_1\tilde{u})}{(\tilde{u}+c\tilde{v}+\beta_1)^2}-m_2-\frac{bc_1\tilde{w}}{(\tilde{v}+b)^2}\right]>\left[\frac{-(\tilde{u}^2+\beta_1\tilde{u})}{(\tilde{u}+c\tilde{v}+\beta_1)^2}\right](\quad (13)$$

$$\left[\frac{\alpha_1(\tilde{u}^2+\beta_1\tilde{u})}{(\tilde{u}+c\tilde{v}+\beta_1)^2}-m_2-\frac{bc_1\tilde{w}}{(\tilde{v}+b)^2}\right]\left[m_3\tilde{u}-m_4+\frac{m_5\tilde{v}}{(\tilde{v}+b)}\right]>\frac{-c_1\tilde{v}}{(\tilde{v}+b)} \quad (15)$$

then $E_4=(\tilde{u},\tilde{v},\tilde{w})$ is GAS.

Proof: Characterize a Lyapunov function

$$V=\left(\tilde{u}-\tilde{u}^*-\tilde{u}^*-\tilde{u}^*\ln\frac{\tilde{u}}{\tilde{u}^*}\right)+\left(\tilde{v}-\tilde{v}^*-\tilde{v}^*-\tilde{v}^*\ln\frac{\tilde{v}}{\tilde{v}^*}\right)+\left(\tilde{w}-\tilde{w}^*-\tilde{w}^*-\tilde{w}^*\ln\frac{\tilde{w}}{\tilde{w}^*}\right)$$

Taking we differentiate w.r.t. t along the system solution, we get

$\frac{dV}{dt}=-c_{11}(\tilde{u}-\tilde{u}^*)^2-c_{22}(\tilde{p}-\tilde{p}^*)^2-c_{33}(\tilde{w}-\tilde{w}^*)^2+c_{12}(\tilde{u}-\tilde{u}^*)(\tilde{v}-\tilde{v}^*)+c_{23}(\tilde{v}-\tilde{v}^*)(\tilde{w}-\tilde{w}^*)$.

Where, $c_{11}=1-2\tilde{u}-\frac{c\tilde{v}^2+\beta_1\tilde{v}}{(\tilde{u}+c\tilde{v}+\beta_1)^2}-\tilde{w}, c_{12}=\frac{-(\tilde{u}^2+\beta_1\tilde{u})}{(\tilde{u}+c\tilde{v}+\beta_1)^2}, c_{13}=0$

$$c_{21}=\frac{\alpha_1(\tilde{v}^2+\beta_1\tilde{v})}{(\tilde{u}+c\tilde{v}+\beta_1)^2}, c_{22}=\left[\frac{\alpha_1(\tilde{u}^2+\beta_1\tilde{u})}{(\tilde{u}+c\tilde{v}+\beta_1)^2}-m_2-\frac{bc_1\tilde{w}}{(\tilde{v}+b)^2}\right], c_{23}=\frac{-c_1\tilde{v}}{(\tilde{v}+b)}, y_{31}=m_3\tilde{w}, c_{32}=\frac{bm_5\tilde{w}}{(\tilde{v}+b)^2}c_{33}$$
$$=m_3\tilde{u}-m_4+\frac{m_5\tilde{v}}{(\tilde{v}+b)}$$

if the following inequalities hold:

$$c_{11}>0, c_{22}>0, c_{33}>0, \quad (16)$$
$$c_{12}^2<c_{11}c_{22}, \quad (17)$$
$$c_{23}^2<c_{22}c_{33} \quad (18)$$

It is easy to see that all the conditions are satisfied. Biologically, the boundedness and stability of a system show that the model system is well mannered. Furthermore, it implies that no species grows exponentially for a long period.

Hopf-bifurcation of Non-spatial System

Here, we have established a theorem that clarifies Hopf bifurcation, where c stands for the bifurcation parameter.

Theorem 7: Hopf-bifurcation occurs in model system (1)-(3), When c, crosses a threshold value c', near $E_4=(\tilde{u},\tilde{v},\tilde{w})$ if conditions hold as: $A(c)>0, C(c)>0, A(c')B(c')-C(c')=0$ and $[A(c')B(c')]'\neq C'(c')$.

Proof: It is easy to see that E_4 is LAS. If threshold value c' exist s.t.

$A(c')B(c')-C(c')=0$

For $c=c'$ the characteristic equation is as:

$\left(\lambda^2(c')+B(c')\right)\left(\lambda(c')+A(c')\right)=0$

Roots are: $-A(c'), \imath\sqrt{B(c')}$ and $-\imath\sqrt{B(c')}$. If transversality condition $\left.\frac{Re(\lambda(B))}{dc}\right|_{c=c'}\neq 0$ hold, then Hopf-bifurcation occurs at $c=c'$. roots are

$\lambda_1(c)=\mu(c)+\imath\nu(c)$,

$\lambda_2(c)=\mu(c)+\imath\nu(c)$,

$\lambda_3(c) = -A(c)$.

Substituting into we get

$Q(c)\mu'(c) - L(c)v'(c) + M(c) = 0,$
$L(c)\mu'(c) + Q(c)v'(c) + N(c) = 0,$

Where,

$Q(c) = 3u^2(c) + 2A(c)\mu(c) + B(c) - 3v^2(c)$
$L(c) = 6\mu(c)v(c) + 2Av(c)$
$M(c) = \mu^2(c)A'(c) + c'\mu(c) + C'(c) - A'(c)v^2(c)$
$N(c) = 2\mu(c)v(c)A'(c) + B'(c)v(c).$

Here, $\mu(c) = 0, v(c) = \sqrt{B(c')}$, we get

$Q(c') = -2B(c'), L(c') = 2A(c')\sqrt{B(c')}, M(c') = C'(c') - A'(c')B(c'), N(c') = B'(c')\sqrt{B(c')}.$

Solving $\mu'(c)$, we get:

$\frac{Re(\lambda_j(c))}{dc}\Big|_{c=c'} = \mu'(c)_{c=c'} = -\frac{L(c')N(c')+Q(c')M(c')}{Q^2(c')+L^2(c')} = \frac{1}{2}\frac{c'(c')-\left(A(c')B(c')\right)'}{A^2(c')+B(c')} \neq 0$. If $\left(A(c')B(c')\right)' \neq C'(c')$ and $\lambda_3(c') = -A(c') < 0$. Hence the theorem is proved.

Center Manifold Theorem

Center Manifold theorem is applied to check stability if one or more roots are zero and next one is negative real part.Now we shift $E(u_1, v_1, w_1)$ at the origin and get the transformation $u = u + u_1, v = v + v_1, w = w + w_1$ in the system (4)-(6) and we get

$$\frac{du}{dt} = x_{11}u + x_{12}v + x_{13}w + x_{14}u^2 + x_{15}uv + x_{16}v^2 + x_{17}vw + x_{18}w^2 + x_{19}uw$$
$$\frac{dv}{dt} = x_{21}u + x_{22}v + x_{23}w + x_{24}u^2 + x_{25}uv + x_{26}v^2 + x_{27}vw + x_{28}w^2 + x_{29}uw$$
$$\frac{dw}{dt} = x_{31}u + x_{32}v + x_{33}w + x_{34}u^2 + x_{35}uv + x_{36}v^2 + x_{37}vw + x_{38}w^2 + x_{39}uw$$

Here, the coefficients x_{ij} are given in Appendix A. System turns as:

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$$\begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{bmatrix} = \begin{bmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix} + \begin{bmatrix}x_{14}u^2 + x_{15}uv + x_{19}uw\\ x_{24}u^2 + x_{25}uv + x_{27}uw\\ x_{37}vw + x_{39}uw\end{bmatrix}$$

Theorem 8: Eigen values of $J(E_1)$ is a saddle point. Proof: For E_1 the above system reduce to,

$$\begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{bmatrix} = \begin{bmatrix}-1 & \frac{-1}{1+\beta_1} & -1\\ 0 & \frac{\alpha_1}{1+\beta_1} & -m_2\\ 0 & 0 & m_3 - m_4\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix} + \begin{bmatrix}x_{14}u^2 + x_{15}uv + x_{19}uw\\ x_{24}u^2 + x_{25}uv + x_{27}uw\\ x_{37}vw + x_{39}uw\end{bmatrix}$$

E.V. is given as $-1, \frac{\alpha_1}{1+\beta_1} - m_2$ and $m_3 - m_4$. If we consider $\frac{\alpha_1}{1+\beta_1} = m_2$ then the equilibrium point $E_1(1,0,0)$ is non-hyperbolic type.

Now we use the transformation

$$\begin{bmatrix}u\\ v\\ w\end{bmatrix} = R\begin{bmatrix}x\\ y\\ z\end{bmatrix}, \text{ where } R = \begin{bmatrix}1 & -q_{12} & 1\\ 0 & 1 & 0\\ 0 & 0 & q_{33}\end{bmatrix}$$

Then the system turns as,

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & m_3 - m_4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} + \begin{bmatrix}H_1\\ H_2\\ H_3\end{bmatrix}$$

We can see H_i in Appendix B. Now $W^c(0) = (x, y, z)\epsilon R^3: x = g_1(y), z = g_2(y), g_1(0) = 0, g_2(0) = 0, Dg_1(0) = 0, Dg_2(0) = 0$, To calculate $W^c(0)$, let us consider $x = g_1(y) = c_{11}y^2 + c_{12}y^3 + O(\|y\|^4), z = g_2(y) = c_{21}y^2 + c_{22}y^3 + O(\|y\|^4)$. Now we find $c_{11} = 0, c_{12} = N_{13}, c_{21} = 0, c_{22} = 0$. Thus, manifold is shown by $\dot{y} = N_{22}y^2 + N_{21}N_{13}y^4$, it is a saddle point. Hence proved.

Theorem 9: At E_4, the periodic solution will be stable and unstable if $\Lambda < 0$ and $\Lambda > 0$ respectively.

Proof: If Hopf bifurcation exists in the system, it implies that characteristic equations have purely complex conjugates and one negative real root. The roots are given as $-A_1, i\sqrt{A_2}, -i\sqrt{A_2}$. It is non-hyperbolic type roots.

Therefore: $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Where, $\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$, g_{ij} is given in Appendix C.

The system becomes:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & -\sqrt{A_2} & 0 \\ \sqrt{A_2} & 0 & 0 \\ 0 & 0 & -A_1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

We can see AppendiX D to check the values of e_1, e_2 and e_3. Characteristic equations have purely complex conjugate and one negative real root. Therefore, we are unable to find stability, so the Center manifold is given as:

$W^c(0) = (x, y, z)\epsilon R^3: z = g(x, y), |x| < \delta_1, |y| < \delta_2, g(0,0) = 0, Dg(0,0) = 0$. To calculate the center manifold theorem, we consider $z = g(x, y) = a_1x^2 + a_2y^2 + a_3xy + O(\|x\|^3)$, Here we find $a_1 = \frac{1}{A_1}\left(A_{31} - a_3\sqrt{A_2}\right), a_2 = \frac{1}{A_2}\left(A_{32} - a_3\sqrt{A_2}\right)$ and $a_3 = \frac{A_1A_2A_{34} - 2\sqrt{A_2}(A_{32}A_1 - A_{31}A_2)}{(2A_1 + 2A_2 + A_1^2)A_2}$.

Thus, by using the center manifold:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -\sqrt{A_2} \\ \sqrt{A_2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Where $f_1(x, y) = A_{11}x^2 + A_{12}y^2 + A_{14}xy + A_{16}a_1x^3 + A_{15}a_2y^3 + (A_{15}a_1 + A_{16}a_3)x^2y + (A_{15}a_3 + A_{16}a_2)y^2x + O(\|X\|^4), f_2 = A_{21}x^2 + A_{22}y^2 + A_{24}xy + A_{26}a_1x^3 + A_{25}a_2y^3 + A_{25}a_2y^3 + (A_{25}a_1 + A_{26}a_3)x^2y + (A_{25}a_3 + A_{26}a_2)y^2x + O(\|X\|^4)$.

Now, we calculate the First Lyapunov exponent Λ at (0,0,0), to show stability or instability of E_4.

$$\Lambda = \frac{1}{16}\left(f_{1(xxx)} + f_{1(yyy)} + f_{2(xxx)} + f_{2(yyy)}\right) + \frac{1}{16\sqrt{A_2}}\left[f_{1(xy)}\left(f_{1(xx)} + f_{1(yy)}\right) - f_{2(xy)}\left(f_{2(xx)} + f_{2(yy)}\right)\right] - f_{1(xx)}f_{2(xx)} + f_{1(yy)}f_{2(yy)}$$

$$= \frac{1}{16}[6A_{16}a_1 + 2(A_{15}a_3 + A_{16}a_2) + 2(A_{25}a_1 + A_{26}a_3) + 6A_{25}a_3] + \frac{1}{16\sqrt{A_2}}[2A_{14}(A_{11} + A_{12}) - 2A_{24}(A_{21} + A_{22}) - 4A_{11}A_{21} + 4A_{12}A_{22}]$$

Hence theorem is proved.

Numerical Simulation

Here, we will perform numerical analysis supporting the analytical findings. We used ode 45 solver in MATLAB by taking the parameter values as: $\beta_1 = .1$, $c = 0.65, \alpha_1 = .35, m_2 = .25, c_1 = 3.15, b = 0.5, m_3 = .01, m_4 = .5, m_5 = 4.93$. First, we see the outcome of constant c as shown in Figs. 2-6. The diagrams indicate that the system become unstable at $c = 0.4$, as we increase the value of c it tends to Hopf bifurcation at the critical value $c' = 0.37$, to better understand, at different values of c, we observe some of time series and phase portrait diagrams.

Discussions and Conclusions

This work investigates the dynamical nature of system by using Beddington-DeAngelis and Holling type-II functional response to inspect the interaction between prey and predators. It shows a major role to preserve the dynamical system balance. A schematic illustration is provided in Fig. 1 to help visualize the model formulation. The proposed system is investigated using differential equation theory and several dynamical techniques like boundedness, local stability, global stability, and bifurcation. Then, we demonstrated that the solutions are uniformly bounded. We determined equilibrium points and examined their stability. We found a Hopf bifurcation see in Fig. 2-6 with parameter c. As the value of c become larger system become unstable. For advancement, we have given the numerical solutions using MATLAB to validate analytical results for temporal model system.

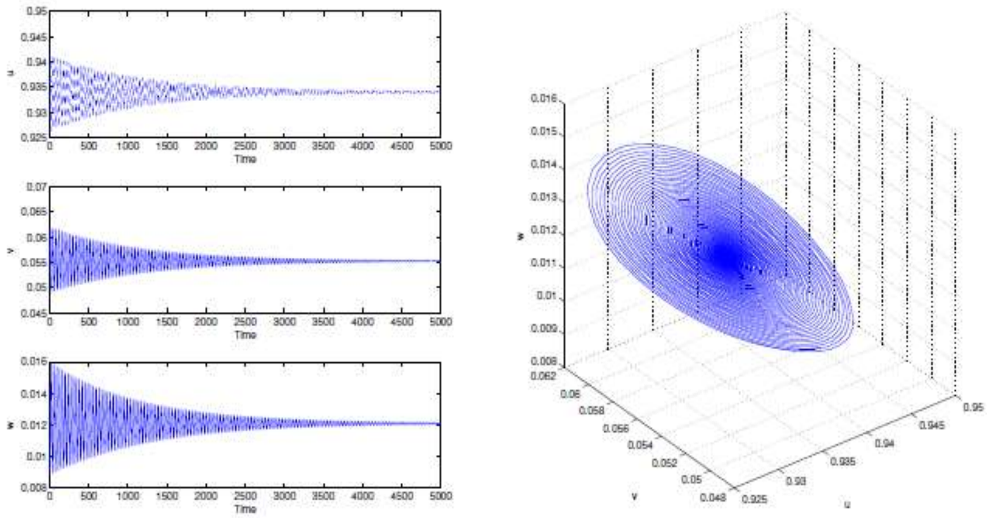

Figure 2: Existence of Hopf bifurcation c = 0.4

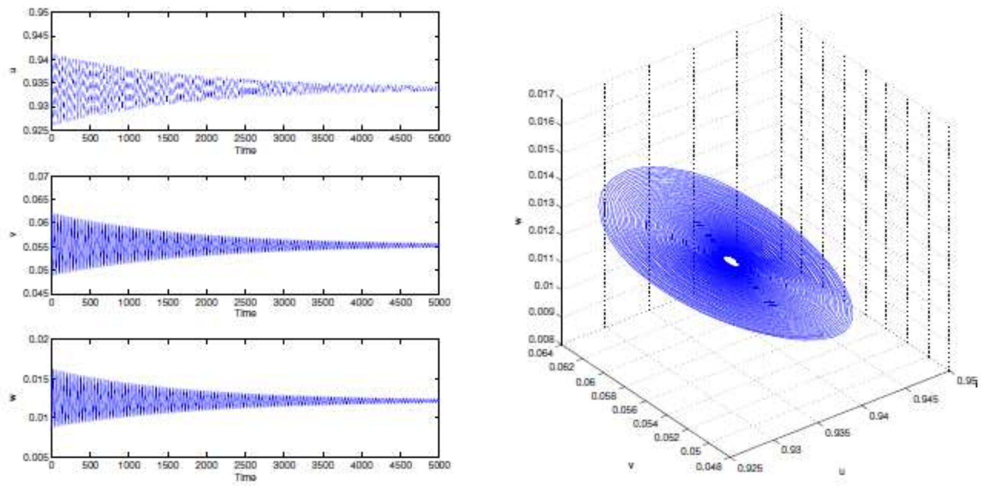

Figure 3: Existence of Hopf bifurcation c = 0.37

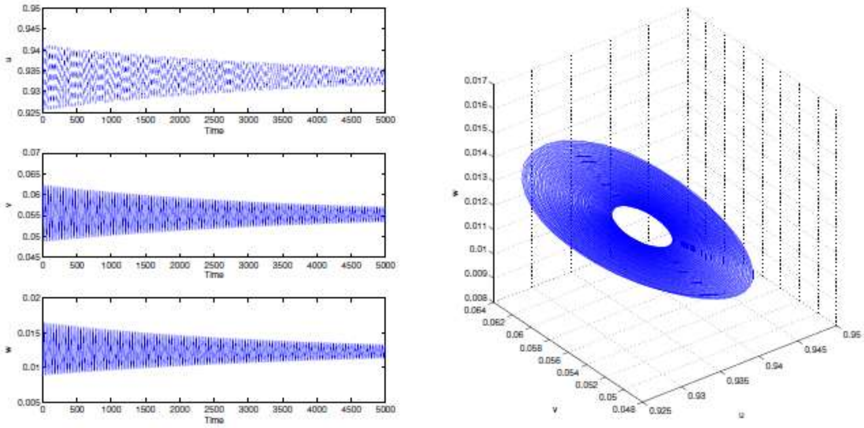

Figure 4: Existence of Hopf bifurcation c = 0.34

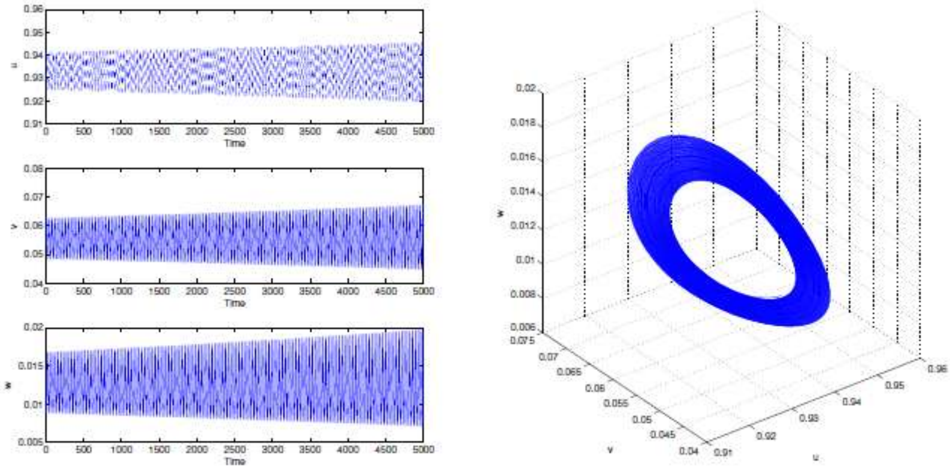

Figure 5: Existence of Hopf bifurcation c = 0.3

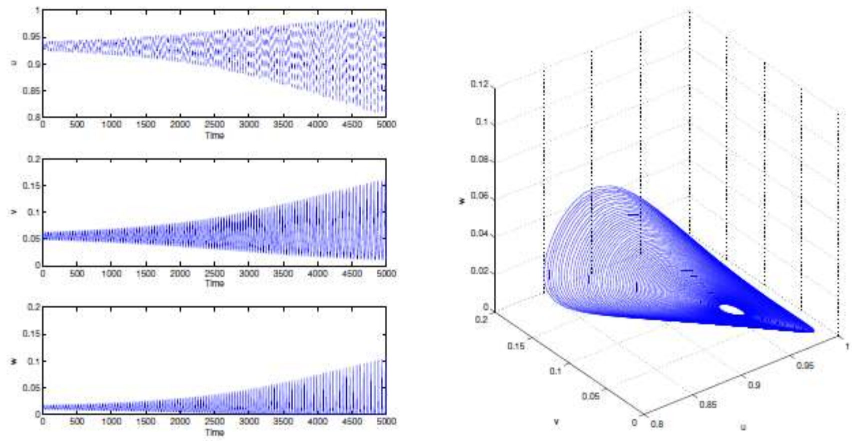

Figure 6: Existence of Hopf bifurcationc = 0.25

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Appendices

- **Appendix A**

$x_{11} = 1 - 2u - \frac{(cv^2+\beta_1 v)}{(u+cv+\beta_1)^2} - w, x_{12} = \left[\frac{-(\tilde{u}^2+\beta_1\tilde{u})}{(\tilde{u}+c\tilde{v}+\beta_1)^2}\right], x_{14} = -2 + \frac{2(cv^2+\beta_1 v)}{(\tilde{u}+c\tilde{v}+\beta_1)^3}, x_{15} = \frac{2(\tilde{u}^2+\beta_1\tilde{u})}{(\tilde{u}+c\tilde{v}+\beta_1)^3} - \frac{(2\tilde{u}+\beta_1)}{(\tilde{u}+c\tilde{v}+\beta_1)^2}, x_{16} = 0, x_{17} = -1, x_{18} = \frac{2c(cu^2+\beta_1 u)}{(u+cv+\beta_1)^3}, x_{19} = 0, x_{21} = \frac{\alpha_1(c\tilde{u}^2+\beta_1\tilde{u})}{(\tilde{u}+c\tilde{v}+\beta_1)^2},$
$x_{22} = \left[\frac{\alpha_1(\tilde{u}^2+\beta_1\tilde{u})}{(\tilde{u}+c\tilde{v}+\beta_1)^2} - m_2 - \frac{bc_1\tilde{w}}{(\tilde{v}+b)^2}\right], x_{23} = \frac{-c_1\tilde{v}}{(\tilde{v}+b)}, x_{24} = \frac{-2\alpha_1(cv^2+\beta_1 v)}{(u+cv+\beta_1)^3}, x_{25} = \frac{-2c\alpha_1(cv^2+\beta_1 v)}{(u+cv+\beta_1)^3} + \frac{\alpha_1(2cv+\beta_1)}{(u+cv+\beta_1)^2}, x_{26} = 0, x_{27} = 0, x_{28} = \frac{-2c\alpha_1(cu^2+\beta_1 u)}{(u+cv+\beta_1)^3} + \frac{(2bcc_1 w)}{(v+b)^3}, x_{29} = 0, x_{31} = m_3\tilde{w},$ $x_{32} = \frac{bm_5\tilde{w}}{(\tilde{v}+b)^2}$
$x_{33} = m_3\tilde{u} - m_4 + \frac{m_5\tilde{v}}{(\tilde{v}+b)}, x_{34} = 0, x_{35} = 0, x_{36} = m_3, x_{37} = 0, x_{38} = \frac{-2bm_5 w}{(v+b)^3}, x_{39} = m_3.$

- **Appendix B**

The values of H_i is as follows:

$H_1 = x^2 + N_{11}xy + M_{12}xz + M_{13}y^2 + M_{14}yz + M_{15}z^2, H_2 = M_{21}xy + M_{22}v^2 + M_{23}yz, H_3 = M_{31}xz + M_{32}yz + M_{33}z^2.$

- **Appendix** C

Here, $g_{11} = -A_2, g_{12} = 0, g_{13} = A_1^2 - x_{23}x_{32}, g_{21} = x_{31}x_{23}, g_{22} = -x_{21}\sqrt{A_2}, g_{23} = x_{31}x_{23} - x_{21}A_1, g_{31} = 0, g_{32} = -x_{31}\sqrt{A_2}, g_{23} = x_{21}x_{32} - x_{31}A_1.$

- **Appendix** D

$e_1 = D_{11}x^2 + D_{12}y^2 + D_{13}z^2 + D_{14}xy + D_{15}yz + D_{16}zx$
$e_2 = D_{21}x^2 + D_{22}y^2 + D_{23}z^2 + D_{24}xy + D_{25}yz + D_{26}zx$
$e_3 = D_{31}x^2 + D_{32}y^2 + D_{33}z^2 + D_{34}xy + D_{35}yz + D_{36}zx$
with, $D_{11} = C_{11}g_{11}^2 + A_{12}g_{11}g_{21},$
$D_{12} = C_{14}g_{22}g_{32},$
$D_{13} = C_{11}g_{13}^2 + C_{12}g_{13}g_{23} + C_{13}g_{13}g_{33} + C_{14}g_{23}g_{33},$
$D_{14} = C_{12}g_{11}g_{22} + C_{13}g_{11}g_{32} + C_{14}g_{21}g_{32}$
$D_{15} = C_{11}g_{13}g_{22} + C_{13}g_{13}g_{32} + C_{14}g_{22}g_{33} + C_{14}g_{23}g_{32},$
$D_{16} = 2C_{11}g_{11}g_{13} + C_{12}g_{11}g_{23} + C_{12}g_{13}g_{21} + C_{13}g_{11}g_{33} + C_{14}g_{21}g_{33}$
The similar expression for D_{2i} and D_{3i} will be obtained only replacing C_{1j} by C_{2j} and C_{3j} respectively.

$$C_{11} = t_{11}x_{14} + t_{12}x_{24}$$
$$C_{12} = t_{11}x_{15} + t_{12}x_{25}$$
$$C_{13} = t_{11}x_{19} + t_{13}x_{39}$$
$$C_{14} = t_{12}x_{27} + t_{13}x_{39}$$

$$C_{21} = t_{21}x_{14} + t_{22}x_{24}$$
$$C_{22} = t_{21}x_{15} + t_{22}x_{25}$$
$$C_{23} = t_{21}x_{19} + t_{23}x_{39}$$
$$C_{24} = t_{22}x_{27} + t_{23}x_{39}$$
$$C_{31} = t_{11}x_{14} + t_{12}x_{24}$$
$$C_{32} = t_{31}x_{15} + t_{32}x_{25}$$
$$C_{33} = t_{31}x_{19} + t_{33}x_{39}$$
$$C_{34} = t_{32}x_{27} + t_{33}x_{39}$$

Where,

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$$A^{-1} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$$

with $t_{11} = \frac{g_{22}g_{33} - g_{23}g_{32}}{|A|}, t_{12} = \frac{g_{32}g_{13}}{|A|}, t_{13} = \frac{-g_{13}g_{22}}{|A|}, t_{21} = \frac{-g_{21}g_{33}}{|A|}, t_{22} = \frac{g_{11}g_{33}}{|A|}, t_{23} = \frac{g_{13}g_{21} - g_{11}g_{23}}{|A|}, t_{31} = \frac{g_{21}g_{32}}{|A|}, t_{32} = \frac{-g_{11}g_{32}}{|A|}, t_{33} = \frac{g_{11}g_{22}}{|A|}$ where, $|A|$ represents the determinant of A..

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