International Journal of Global Research Innovations & Technology (IJGRIT)
e-ISSN : 2583-8717, Impact Factor: (Cosmos:6.972& I2OR: 5.002), Volume 04, No. 01, January-March, 2026, pp 139-143

Generalization of Common Fixed Point Theorems for Probabilistic nearly Densifying Mappings in Probabilistic S-metric Spaces

Rajesh Vyas[1] | Shubham Prajapat[2*]

[1]Professor of Mathematics, St. Paul Institute of Professional Studies, Boundary Lane, Near Lalaram Nagar, Indore, Madhya Pradesh, India.
[2]School of Mathematics, Devi Ahilya University, Takshila campus Khandwa Road, Indore, Madhya Pradesh, India.

*Corresponding Author: shubhamprajapati536@gmail.com

Citation: Vyas, R. & Prajapat, S. (2026). Generalization of Common Fixed Point Theorems for Probabilistic nearly Densifying Mappings in Probabilistic S-metric Spaces. International Journal of Global Research Innovations & Technology, 04(01), 139–143. https://doi.org/10.62823/IJGRIT/4.1.8642

ABSTRACT

In this paper, we establish common fixed point theorems for four self maps using probabilistic nearly densifying mappings and extend the results of Aeshah Hassan Zakri et al. [18] in the framework of probabilistic S- metric spaces. Probabilistic S- metric space is the extended notion of S- metric spaces and Menger probabilistic metric spaces.

***Keywords**: Probabilistic S-Metric Space, Probabilistic Nearly Densifying Mappings.*

***2020 Mathematics Subject Classification**: 47H10, 54H25.*

Introduction

In 1942, Karl Menger [11] introduced the generalization of metric spaces involving probabilistic distance. Main structures of probabilistic metric space defined by B. Schweizer and A. Skalar [14]. Probabilistic S_b-metric spaces introduced by A. Kalpana and M. Saraswathi **(**2020**)** [9]. Kuratowski [10] introduced the notion of non-compactness of bounded subset of a metric spaces. Furi and Vignoli [6] were motivated by the results of Kuratowski and they introduced the notion of densifying or condensing mapping in terms of measure of Kuratowski and produced results in fixed points. Bocsan and Constantin [2] introduced the notion of Kuratowski's measure of non-compactness in Probabilistic metric spaces. Bocsan [3] studied the notion of probabilistic densifying mappings. Later, Hadžić [8], Tan [17], Chamolaet al. [4], Dimri and Pant [5], Pant et al. [12-13] and Singh and Pant [15] proved some results. Ganguly et al. [7] introduced the notion of probabilistic nearly densifying mappings. Aeshah Hassan Zakriet al. [18] proved results for probabilistic nearly densifying mappings.

Preliminaries

- **Definition2.1** [1]Let(X, S, τ) is probabilistic S -metric spaces with a continuous t – norm and A be a nonempty subset of X. Then a function D_A defined by

  $D_A(x) = \sup_{t<x} \{ \inf_{u,v,w \in A} F_{u,v,w}(t)\}$is called the probabilistic diameter of A.
- **Definition 2.2**[14] A binary operation *T:* [0, 1] × [0, 1]→ [0, 1] is a continuous ***t*-norm** ifit satisfies the following conditions:
  - *T* is commutative and associative,
  - *T* is continuous,
  - *T* (*p*, 1) = *p*for all *p*∈ [0, 1],
  - *T* (p, *q*) ≤ *T* (*r*, *s*) when*p*≤ *r*and *q*≤ *s*, and *p, q, r, s*∈ [0, 1].
- **Definition 2.3**[10]For a probabilistic bounded subset A of X, α_A(x) defined by α_A(x) = sup {ε ≥ 0: There exists a finite cover A of A such that D_Q(x) ≥ ε for all Q ∈A} is called Kuratowski's function.

Kuratowski's function meets the following criteria:

- $\alpha_A \in \Delta_+$, the set of distribution function.
- α_A(x) =D_A(x);
- If S ≠ A ⊂ B ⊂ C ⊂ X, then α_A(x) ≥ α_B(x) ≥ α_C(x);
- $\alpha_{A\cup B\cup C}$(x) = min{α_A(x),α_B(x),α_C(x)}
- Let $\bar{A}$ be the closure of A in the (ε, λ) topology on X; then
- $\alpha_{\bar{A}}$(x) = α_A(x)
- A is probabilistic precompact (totally bounded) if α_A = H,

Where H is defined as distribution function.

$$H(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$$

- **Definition 2.4**[18] Let K be a family of self mappings in X. A subset Z of X is said to be K-invariant if kZ ⊆ Z for all k ∈ K.
- **Definition 2.5**[18] Let K^* be the semigroup generated by K under composition *.

  {k^* : n ≥ 0} ⊆K^* for any k ∈ K and K^* (u) = {u} ∪ {ku : k ∈K^* } for any u ∈ X.
- **Definition2.6**Let (X, S, T) be a probabilistic S- metric space. A continuous mapping f of X into itself is said to be probabilistic densifying mapping if and only if, for every subset of X, $\alpha_A < H$ implies $\alpha_{f(A)}$>α_A.
- **Definition 2.7** A self mapping f of X into itself is said to be probabilistic nearly densifying mapping if $\alpha_{f(A)}$>α_A, whenever $\alpha_A < H$, A ⊂ H and A is f- invariant.
- **Definition 2.8** [14] The set of all distance distribution functions is denoted by ∆ and distribution functions which satisfy F (0) = 0 are called Δ_+.

Distribution function *F: (−∞,* +∞) → [0*,* 1], is left-continuous at every real point, non-decreasing and satisfies *F (−∞)* = 0 and *F (*+∞*)* = 1.

Distance distribution function is given by $\varepsilon_0(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$

- **Definition 2.9**[16]S-metric spaces**:** Let X be a nonempty set, pair (X, S) be S-metric space.S-metric on X is a function *S*: X^3 → [0*,* ∞) that satisfies the following properties, for each p, q, r, a∈*X*;
  - S (p, q, r) ≥ 0;
  - S (p, q, r) = 0 if and only if p = q = r;
  - S (p, q, r) ≤ S (p, p, a) + S (q, q, a) + S (r, r, a).
- **Definition 2.10 Probabilistic S-metric spaces:**Let X is a nonempty set, τ is a triangle function and $\Delta_+$is a probability distribution function, then S:X×X×X→$\Delta_+$is a mapping satisfies the following,
  - $S_{(p,p,p)}=\varepsilon_0$
  - If p ≠ q then $S_{(p,p,q)} \neq \varepsilon_0$
  - If$S_{(p,q,r)} \geq \tau\ (S_{(p,p,s)}, S_{(q,q,s)}, S_{(r,r,s)})$

  Distance distribution function is given by $\varepsilon_0(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$

  for all p, q, r, s ∈ X. Then (X, S, τ) is called an extended notion of probabilistic S -metric spaces.

Main results

- **Theorem 1.** Let A, B, C and D be four continuous and nearly densifying self – mapping on a complete probabilistic S- metric spaces (S, X, T) where sup x × x × x =1 and D commutes with A, B and C. If for all x < 1, u, v, w ∈X, the following conditions are satisfied,

$$S_1(Au, Bv, Cw) > \min\{S_2(Du, Dv, Dw), S_2(Du, Au, Au), S_1(Dv, Bv, Bv), S_3(Dw, Cw, Cw), \frac{S_1(Dv,Bv,Bv)S_2(Du,Au,Au)S_3(Dw,Cw,Cw)}{S_2(Du,Dv,Dw)}\} \quad (1)$$

For Du ≠Dv ≠ Dw, Au ≠ Bv ≠ Cw

S_2(Au, Cv, Bw) > min {S_3(Du, Dv, Dw),S_3(Du, Au, Au), S_2(Dv, Cv, Cv),S_1(Dw, Bw, Bw), $\frac{S_1(Dw,Bw,Bw)S_2(Dv,Cv,Cv)S_3(Du,Au,Au)}{S_3(Du,Dv,Dw)}$} (2)

For Du ≠ Dv ≠ Dw, Au ≠ Cv ≠Bw

S_3(Cu, Bv, Aw) > min {S_1(Du, Dv, Dw),S_1(Du, Cu, Cu), S_3(Dv, Bv, Bv),S_2(Dw, Aw, Aw), $\frac{S_1(Du,Cu,Cu)S_2(Dw,Aw,Aw)S_3(Dv,Bv,Bv)}{S_1(Du,Dv,Dw)}$} (3)

For Du ≠ Dv ≠ Dw, Cu ≠ Bv ≠Aw

Where S_1, S_2, S_3 are real valued mappings from X×X×X to Δ_+ a collection of all distribution functions with any of S_1, S_2 or S_3 being upper semicontinuous then A and D or B and D or C and D have coincidence point.

Proof. For x_0∈ X, let F = K(x_0) and E = $ABCD$

F = $\{x_0\}$∪ A(F)∪ B(F)∪ C(F) ∪ D(F)

If α_F< H then $\alpha_F = \alpha_{\{x_0\}\cup A(F)\cup B(F)\cup C(F)\cup D(F)}$

min{$\alpha_{\{x_0\}\cup A(F)\cup B(F)\cup C(F)\cup D(F)}$}>$\alpha_F$

Which contradicts assumption, hence $\bar{F}$ is precompact.

Let G = $\bigcap_{n=0}^{\infty}(ABCD)^n(\bar{F})$

Then EG = G, G is nonempty compact subset of E. The continuous nature of A, B, C, and D results in them following that.

A$\bar{F} \subset \bar{F}$, $B\bar{F} \subset \bar{F}, C\bar{F} \subset \bar{F}$, $D\bar{F} \subset \bar{F}$

A$(\bar{G}) \subset \bar{G}, B(\bar{G}) \subset \bar{G}, C(\bar{G}) \subset \bar{G}, D(\bar{G}) \subset \bar{G}$

Now D(G) = $\bigcap_{n=0}^{\infty} D(ABCD)^n(\bar{F}) \subseteq \bigcap_{n=0}^{\infty}(ABCD)^n D(\bar{F}) \subseteq$ G

G = ABCD(G) = DABC(G) ⊂ $DAB(G)$ ⊂DA(G) ⊂ $D(G)$

Which implies D(G) = G

Let S_1 is upper semicontinuous then T : G → τ defined by T(u) = $S_1(Du, Bu, Bu)$

is upper semi continuous. Let T assumes its maximal value of same point q in G. If q ∈D^2(G) so there is m ∈ G such that q = D^2(m)

Suppose neither A and D or B and D or C and D have coincidence point then

T(ABC(m)) = $S_1(DABC(m), B^2AC(m), B^2AC(m))$

= $S_1(ADBC(m), B^2AC(m), CB^2A(m))$

>min {S_2(DDBC(m), DABC(m), D$B^2A(m)$),S_2(DDBC(m), ADBC(m), ADBC(m)), S_1(DABC(m), BABC(m), BABC(m)),S_3(D$B^2A(m)$, C$B^2A(m)$, C$B^2A(m)$),

$\frac{S_1(DABC(m), BABC(m), BABC(m))S_2(DDBC(m), ADBC(m), ADBC(m))S_3(DB^2A(m), CB^2A(m), CB^2A(m))}{S_2(DDBC(m), DABC(m), DB^2A(m))}$}

= S_2(DDBC(m), DABC(m), D$B^2A(m)$)

T(ABD(m)) = $S_2(D^2AB(m), CABD(m), CABD(m))$

= $S_2(AD^2B(m), CABD(m), BCAD(m))$

> min{S_3(D^3B(m), D^2BA(m),D^2CA(m)),S_3(D^3B(m), AB$D^2(m)$, AB$D^2(m)$), S_2($D^2BA(m)$, CBAD(m), $B^2AD(m)$),S_1(D^2CA(m), BCAD(m), BCAD(m)),

$\frac{S_1(D^2CA(m),BCAD(m),BCAD(m))S_2(D^2BA(m),CBAD(m),B^2AD(m))\,S_3(D^3B(m),ABD^2(m),ABD^2(m))}{S_3(D^3B(m),D^2BA(m),D^2CA(m))}$}

= S_3(D^3B(m), D^2BA(m),D^2CA(m))

T(ACD(m)) = S_3(D^2AC(m), BACD(m), BACD(m))

=S_3(CD^2A(m), BACD(m), BACD(m))

> min {S_1($D^3A(m)$, $D^2AC(m)$,$D^2BC(m)$),S_1($D^3A(m)$, $D^2CA(m)$, $D^2CA(m)$), S_3($D^2AC(m)$, BACD(m), BACD(m)),S_2($D^2BC(m)$, ABCD(m), ABCD(m)),

$$\frac{S_1\left(D^3A(m),D^2CA(m),D^2CA(m)\right)S_2\left(D^2BC(m),ABCD(m),ABCD(m)\right)S_3\left(D^2AC(m),BACD(m),BACD(m)\right)}{S_1\left(D^3A(m),D^2AC(m),D^2BC(m)\right)}\}$$

= $S_1(D^3A(m),\ D^2AC(m),D^2BC(m))$

= $S_1(ADq,\ ACq,BCq)$ = T(q)

Which contradicts q's choice. Therefore, A and D or B and D or C and D have coincidence point.This result holds the same if S_2 is upper semicontinuous.

This completes the proof of the theorem.

- **Theorem 2. (Uniqueness)** Let A, B, C and D be as in theorem 1 satisfying (1), (2) and (3) have coincidence point m then Dm is unique common fixed point of A, B, C and D.

Proof. We have Am = Bm = Cm = Dm

By commutative property of D with A, B and C,

ABD(m) = BAD(m) = DDD(m) and BCD(m) = CBD(m) = DDD(m) or

CAD(m) = ACD(m) = DDD(m) or

ABD(m) = BCD(m) = CAD(m) = DDD(m)

Let D^3m ≠D^2m ≠ Dm then

$S_1(D^3$m,D^2m, Dm) = S_1(AD^2m, BDm, Cm)

> min {$S_2(DD^2$m, DDm, Dm),S_2(DD^2m, AD^2m, AD^2m), S_1(DDm, BDm, BDm),S_3(Dm, Cm, Cm), $\frac{S_1(\text{DDm,BDm,BDm})S_2(\text{D}D^2\text{m,A}D^2\text{m,A}D^2\text{m})S_3(\text{Dm,Cm,Cm})}{S_2(DD^2\text{m,DDm,Dm})}$}

= $S_2(DD^2$m, DDm, Dm) = $S_2(AD^2$m, CDm, Bm)

> min {S_3(DD^2m, DDm, Dm),S_3(DD^2m, AD^2m, AD^2m), S_2(DDm, CDm, CDm),S_1(Dm, Bm, Bm), $\frac{S_1(\text{Dm,Bm,Bm})S_2(\text{DDm,CDm,CDm})S_3(\text{D}D^2\text{m,A}D^2\text{m,A}D^2\text{m})}{S_3(\text{D}D^2\text{m,DDm,Dm})}$}

= S_3(DD^2m, DDm, Dm) = S_3(CD^2m, BDm, Am)

> min {S_1(DD^2m, DDm, Dm),S_1(DD^2m, CD^2m, CD^2m), S_3(DDm, BDm, BDm),S_2(Dm, Am, Am), $\frac{S_1(\text{D}D^2\text{m,C}D^2\text{m,C}D^2\text{m})S_2(\text{Dm,Am,Am})S_3(\text{DDm,BDm,BDm})}{S_1(\text{D}D^2\text{m,DDm,Dm})}$ }

= S_1(DD^2m, DDm, Dm)

Which contradicts our assumption. Hence D^3m = D^2m = Dm.

Thus, Dm is fixed point of D and Dm = D^2m = D^3m = ABD(m) = BCD(m) = CAD(m). Therefore, Dm is common fixed point of A, B and C.

The results of Theorem 11 ofAeshah Hassan Zakri et al. [18] are extended by the above theorem.

Example: Let X is a nonempty set, τ is a triangle function and $\Delta_+$is a probability distribution function, then S:X×X×X→$\Delta_+$is a mapping satisfies

$S_{(p,q,r)}$(t) = $\frac{t}{t+|p-q|+|q-r|+|r-s|}$ such that t **>**0

Where a function f(x) = $\frac{x}{3}$defined as densifying mapping if and only if, for every subset of X, $\alpha_A <$ H implies $\alpha_{f(A)}$>α_A. In term, $|p-q|$ = $\left|\frac{p}{3}-\frac{q}{3}\right|$ =$\frac{1}{3}|p-q|$.

Lemma: Let X is a nonempty set, τ is a triangle function and $\Delta_+$is a probability distribution function, then S:X×X×X→Δ_+. Where, (X, S, τ) is called probabilistic S -metric spaces.

Let A and B bounded subsets of X. The function α is the Kuratowski measure of noncompactness. Then α (A ∪ B) = min {α (A), α (B)}

α (A) = 0 implies that A is probabilistic precompact (totally bounded) if α (A) = α ($\bar{A}$).

Conclusion

Using probabilistic nearly densifying mappings in probabilistic S-metric spaces, we proved common fixed point theorems for four self maps and uniqueness of their fixed points, drawing inspiration from the results of Aeshah Hassan Zakri et al. [18].We have shown terminology with example, and a lemma that is crucial to the paper.

References

1. Ayaz Ahmad, “Probabilistic diameter and its properties”,International Journal of Mathematics and Statistics Invention (IJMSI),E-ISSN: 2321 – 4767 P-ISSN: 2321 – 4759 Volume 3 Issue 7,PP 40- 44.
2. G. Bocsan and G. Constantin, “The Kuratowski function and some applications to the probabilistic metric spaces,” Attidella Accademia NazionaledeiLincei. Rendiconti. Classe di ScienzeFisiche, Matematiche e Naturali, vol. 55,no. 8, pp. 236–240, (**1973)**.
3. G. Bocsan, “On some fixed point theorems in probabilistic metric spaces,” Mathematica Balkanica, vol. 4, pp. 67–70, (**1974)**.
4. K. P. Chamola, B. D. Pant, and S. L. Singh, “Common fixed point theorems for probabilistic densifying mappings,” Mathematica Japonica, vol. 36, no. 4, pp. 769–775, 1991.
5. R. C. Dimri and B. D. Pant, “Fixed points of probabilistic densifying mappings,” Journal of Natural & Physical Sciences, vol. 16, no. 1-2, pp. 69–76, 2002.
6. M. Furi and A. Vignoli, “Fixed points for densifying mappings,”AccademiaNazionaledeiLincei, vol. 47, pp. 465–467, **1969**.
7. A. Ganguly, A. S. Rajput, and B. S. Tuteja, “Fixed points of probabilistic densifying mappings,” Indian Academy of Mathematics: Journal, vol. 13, no. 2, pp. 110–114, 1991.
8. O. Hadžić, “Fixed point theorems for multivalued mappings in ´ probabilistic metric spaces,” Fuzzy Sets and Systems, vol. 88, no. 2, pp. 219–226, 1997
9. A. Kalpana and M, Saraswathi, Probabilistic S_b- metric spaces, Malaya Journal of Matematik, Vol. 8, No. 3, 920-923, 2020.
10. C. Kuratowski, “Sur les espaces complets,” Fundamenta Mathematicae, vol. 15, pp. 301–309, (**1930)**.
11. K. Menger, “Statistical metrics,” Proceedings of the National Academy of Sciences of the United States of America, vol. 28, pp. 535–537, 1942.
12. B. D. Pant, B. M. L. Tiwari, and S. L. Singh, “Common fixed point theorems for densifying mappings in probabilistic metric spaces,” Honam Mathematical Journal, vol. 5, pp. 151–154, 1983.
13. B. D. Pant, R. C. Dimri, and V. B. Chandola, “Some results on fixed points of probabilistic densifying mappings,” Bulletin of the Calcutta Mathematical Society, vol. 96, no. 3, pp. 189–194, 2004.
14. B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J.Math,10(**1960**)313-334. *http://dx.doi.org/10.1137/1107042*
15. S. L. Singh and B. D. Pant, “Common fixed points of a family of mappings in Menger and uniform spaces,” Rivista di Matematica della Universita di Parma: Serie IV `, vol. 14, no. 4, pp. 81– 85, 1988.
16. S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixedpoint theorems in S-metric spaces. MatVesn. 64(3), 258-266(2012).
17. D. H. Tan, “On probabilistic condensing mappings,” Revue Roumaine de MathematiquePures et Appliqu ´ ees ´, vol. 26, no. 10, pp. 1305–1317, 1981.
18. Aeshah Hassan Zakri, Sumitra Dalal, Sunny Chauhan, Jelena Vujaković, “Common Fixed Point Theorems for Probabilistic Nearly Densifying Mappings”, Abstract and Applied Analysis, vol. 2015, Article ID 497542, 5 pages, 2015.